

Rigidity theorems of complete Kähler-Einstein manifolds and complex space forms

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We derive some elliptic differential inequalities from the Weitzenböck formulas for the traceless Ricci tensor of a Kähler manifold with constant scalar curvature and the Bochner tensor of a Kähler-Einstein manifold respectively. Using elliptic estimates and maximum principle, some L^p and L^∞ pinching results are established to characterize Kähler-Einstein manifolds among Kähler manifolds with constant scalar curvature, and others are given to characterize complex space forms among Kähler-Einstein manifolds. Finally, these pinching results may be combined to characterize complex space forms among Kähler manifolds with constant scalar curvature.

1 Introduction

One of the major problems in geometry is to investigate the rigidity phenomena of some canonical geometric structures on manifolds. Various geometric invariants (tensors or quantities) have been introduced to measure the deviation of a general structure from some canonical one. For a Riemannian manifold, the traceless Ricci tensor measures its deviation from an Einstein manifold, while the Weyl curvature tensor measures its deviation from a conformal flat manifold. These tensors have been used to establish some rigidity theorems for some special Riemannian manifolds (cf. [HV], [IS], [Ki], [PRS], [Sh1,2], etc.).

Over the past decades, much effort has been made to establish the existence of Kähler metrics with constant scalar curvature on a compact Kähler manifold (cf. [Ti], [Do], [LS], [Ch] and the references therein). Among these metrics, Kähler-Einstein metrics form a notable subclass, which plays an important role in both complex geometry and physics. Besides the existence, the uniqueness and rigidity of these canonical Kähler metrics are also important for geometric applications. Back to early 50's, Calabi had proved the uniqueness for Kähler-Einstein metrics with nonpositive scalar

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curvature. In 1986, Bando and Mabuchi [BM] showed the uniqueness for Kähler-Einstein metrics with positive scalar curvature. Actually their uniqueness results were established within a given Kähler class. In [IN], Itoh and Nakagawa obtained some local rigidity results of a Kähler-Einstein metric in the moduli space of Einstein metrics by means of the variational stability. On the other hand, complete noncompact Kähler-Einstein manifolds have also received much attention (cf. [TY1], [TY2], [Ku]).

In this paper, we will consider complete Kähler manifolds with constant scalar curvature and investigate the following two problems:

(A) The rigidity of Kähler-Einstein metrics among Kähler metrics with constant scalar curvature;

(B) The rigidity of Kähler metrics with constant holomorphic sectional curvature among Kähler-Einstein metrics.

As in the real case, we may use the traceless Ricci tensor E to measure the deviation of a Kähler metric from a Kähler-Einstein metric. In 1949, Bochner introduced the so-called Bochner curvature tensor B on a Kähler manifold, which is an analogue of the Weyl curvature tensor. It seems a little tautological to say that the Bochner curvature tensor measures the deviation of a Kähler metric from a Bochner flat Kähler metric. Nevertheless, for a Kähler-Einstein metric, the Bochner curvature tensor measures directly the difference of its curvature tensor from that of the Kähler metric with constant holomorphic sectional curvature (see (4.1) in §4). In order to study Problems A and B, we will derive the Weitzenböck formulas for $|E|^2$ and $|B|^2$ respectively. First, note that if the scalar curvature is constant, then E is a $(1, 1)$ -type Codazzi tensor. Next, if the metric is Kähler-Einstein, then B satisfies the second Bianchi identity, which exhibits a Codazzi type property too. These properties for E and B , combined with some refined Kato inequalities, enable us to deduce some differential inequalities for $|E|$ and $|B|$ respectively. We will treat these differential inequalities on both complete noncompact and compact Kähler manifolds by means of elliptic estimates and maximum principle. Some L^p and L^∞ pinching results will be established to characterize Kähler-Einstein manifolds among complete Kähler manifolds with constant scalar curvature, and others will be given to characterize complex space forms among Kähler-Einstein manifolds. Consequently we may also characterize complex space forms among complete Kähler manifolds with constant scalar curvature. Finally, we would like to mention that the authors [DLR] have established similar results for complete Sasakian manifolds.

2 Preliminaries

Let (M, g) be a smooth Riemannian n -manifold with dimension $n \geq 2$ and let S_g denote the scalar curvature of g . The Yamabe constant is defined by

$$\Lambda(M, g) = \inf_{0 \neq u \in C_0^\infty(M)} \frac{\int_M (|\nabla u|^2 + \frac{n-2}{4(n-1)} S_g u^2) dV_g}{(\int_M |u|^{\frac{2n}{n-2}} dV_g)^{\frac{n-2}{n}}}.$$

If $\Lambda(M, g) > 0$, then one has the following Sobolev type inequality

$$\Lambda(M, g) (\int_M |u|^{\frac{2n}{n-2}} dV_g)^{\frac{n-2}{n}} \leq \int_M (|\nabla u|^2 + \frac{n-2}{4(n-1)} S_g u^2) dV_g \quad (2.1)$$

for any $u \in C_0^\infty(M)$. It is known that if (M, g) is compact, then the sign of $\Lambda(M, g)$ is basically determined by the sign of the scalar curvature in a conformal class (cf. [He1]). However, there are complete noncompact Riemannian manifolds with both negative scalar curvature and positive Yamabe constant (cf. [SY], [He2]).

From now on, we assume that (M^m, g, J) is a Kähler manifold with complex dimension $m \geq 2$ and $\Lambda(M, g) > 0$. Let (z_α) be a system of local complex coordinates on M and let $g_{\alpha\bar{\beta}}$ ($1 \leq \alpha, \beta \leq m$) be the components of the Kähler metric in the coordinates. The inverse matrix of $(g_{\alpha\bar{\beta}})$ is denoted by $(g^{\alpha\bar{\beta}})$. Let $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ and $R_{\alpha\bar{\beta}}$ denote the components of the curvature tensor and the Ricci tensor respectively. As usual, we will use the summation convention on repeating indices. The complex scalar curvature is defined by

$$R = g^{\alpha\bar{\beta}} R_{\alpha\bar{\beta}}.$$

Note that $S_g = 2R$. In this circumstance the Sobolev inequality (2.1) becomes

$$\Lambda(M, g) (\int_M |u|^{\frac{2m}{m-1}} dV_g)^{\frac{m-1}{m}} \leq \int_M (|\nabla u|^2 + \frac{m-1}{2m-1} R u^2) dV_g \quad (2.2)$$

for any $u \in C_0^\infty(M)$.

In [Bo], S. Bochner introduced the Bocher curvature tensor as follows:

$$\begin{aligned} B_{\alpha\bar{\beta}\gamma\bar{\delta}} = & R_{\alpha\bar{\beta}\gamma\bar{\delta}} + \frac{1}{m+2} (R_{\alpha\bar{\beta}} g_{\gamma\bar{\delta}} + R_{\gamma\bar{\beta}} g_{\alpha\bar{\delta}} + g_{\alpha\bar{\beta}} R_{\gamma\bar{\delta}} + g_{\gamma\bar{\beta}} R_{\alpha\bar{\delta}}) \\ & - \frac{R}{(m+1)(m+2)} (g_{\alpha\bar{\beta}} g_{\gamma\bar{\delta}} + g_{\gamma\bar{\beta}} g_{\alpha\bar{\delta}}) \end{aligned} \quad (2.3)$$

which may be regarded as a complex analogue of the Weyl curvature tensor. Clearly the Bochner tensor B has the same algebraic symmetries as the curvature tensor of a Kähler metric. These includes

$$B_{\alpha\bar{\beta}\gamma\bar{\delta}} = B_{\gamma\bar{\beta}\alpha\bar{\delta}} = B_{\alpha\bar{\delta}\gamma\bar{\beta}}, \quad \overline{B_{\alpha\bar{\beta}\gamma\bar{\delta}}} = B_{\beta\bar{\alpha}\delta\bar{\gamma}}. \quad (2.4)$$

In addition, it has the following metric contraction property:

$$g^{\alpha\bar{\beta}} B_{\alpha\bar{\beta}\gamma\bar{\delta}} = 0. \quad (2.5)$$

The traceless Ricci tensor

$$E = E_{\alpha\bar{\beta}}dz^\alpha \otimes dz^{\bar{\beta}} + E_{\bar{\alpha}\beta}dz^{\bar{\alpha}} \otimes dz^\beta$$

is defined by

$$E_{\alpha\bar{\beta}} = R_{\alpha\bar{\beta}} - \frac{R}{m}g_{\alpha\bar{\beta}}.$$

Then the Bochner curvature tensor may also be expressed as

$$\begin{aligned} B_{\alpha\bar{\beta}\gamma\bar{\delta}} = & R_{\alpha\bar{\beta}\gamma\bar{\delta}} + \frac{1}{m+2}(E_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + E_{\gamma\bar{\beta}}g_{\alpha\bar{\delta}} + g_{\alpha\bar{\beta}}E_{\gamma\bar{\delta}} + g_{\gamma\bar{\beta}}E_{\alpha\bar{\delta}}) \\ & + \frac{R}{m(m+1)}(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\gamma\bar{\beta}}g_{\alpha\bar{\delta}}). \end{aligned} \quad (2.6)$$

For a Kähler manifold, the second Bianchi identity is reduced to

$$R_{\alpha\bar{\beta}\gamma\bar{\delta},\lambda} = R_{\alpha\bar{\beta}\lambda\bar{\delta},\gamma} \quad \text{and} \quad R_{\alpha\bar{\beta}\gamma\bar{\delta},\bar{\lambda}} = R_{\alpha\bar{\beta}\gamma\bar{\lambda},\bar{\delta}}. \quad (2.7)$$

By contracting the indices α and $\bar{\beta}$ in (2.7), we get

$$R_{\gamma\bar{\delta},\lambda} = R_{\lambda\bar{\delta},\gamma} \quad \text{and} \quad R_{\gamma\bar{\delta},\bar{\lambda}} = R_{\gamma\bar{\lambda},\bar{\delta}}. \quad (2.8)$$

A $(1,1)$ -type tensor is called Hermitian symmetric if the matrix of its components is Hermitian symmetric. An Hermitian symmetric $(1,1)$ -type tensor field with the properties (2.8) will be called a $(1,1)$ -type Codazzi tensor. Clearly the Ricci tensor field is a $(1,1)$ -type Codazzi tensor. Thus, if the scalar curvature R is constant, then the traceless Ricci tensor field E is also a $(1,1)$ -type Codazzi tensor.

The usual Ricci identity for commuting covariant derivatives gives

$$E_{\alpha\bar{\beta},\lambda\bar{\mu}} - E_{\alpha\bar{\beta},\bar{\mu}\lambda} = E_{\gamma\bar{\beta}}R_{\gamma\alpha\lambda\bar{\mu}} + E_{\alpha\bar{\gamma}}R_{\gamma\bar{\beta}\lambda\bar{\mu}} \quad (2.9)$$

and

$$B_{\alpha\bar{\beta}\gamma\bar{\delta},\lambda\bar{\mu}} - B_{\alpha\bar{\beta}\gamma\bar{\delta},\bar{\mu}\lambda} = B_{v\bar{v}\gamma\bar{\delta}}R_{v\alpha\lambda\bar{\mu}} + B_{\alpha\bar{v}\gamma\bar{\delta}}R_{v\bar{\beta}\lambda\bar{\mu}} + B_{\alpha\bar{\beta}v\bar{\delta}}R_{v\gamma\lambda\bar{\mu}} + B_{\alpha\bar{\beta}\gamma\bar{v}\bar{\delta}}R_{v\bar{\delta}\lambda\bar{\mu}}. \quad (2.10)$$

We will need the following two lemmas. The first one is an algebraic inequality.

Lemma 2.1. (*[Ok]*) Let λ_α , $\alpha = 1, \dots, m$, be real numbers. If $\sum_{\alpha=1}^m \lambda_\alpha = 0$, then

$$\left| \sum_{\alpha=1}^m \lambda_\alpha^3 \right| \leq \frac{m-2}{\sqrt{m(m-1)}} \left(\sum_{\alpha=1}^m \lambda_\alpha^2 \right)^{3/2}.$$

The next one is a gap result for solutions of an elliptic differential inequality.

Lemma 2.2. (*[PRS]*) Let (M, g) be a complete Riemannian n -manifold on which the following Euclidean-type Sobolev inequality

$$C(n) \left(\int_M |u|^{\frac{2n}{n-2}} dV_g \right)^{\frac{n-2}{n}} \leq \int_M |\nabla u|^2 dV_g \quad (2.11)$$

holds for every $u \in C_0^\infty(M)$ with a positive constant $C(n) > 0$. Suppose that $\psi \in Lip_{loc}(M)$ is a nonnegative solution of

$$\psi \Delta \psi + q(x) \psi^2 \geq A |\nabla \psi|^2 \quad (\text{weakly}) \text{ on } M$$

satisfying

$$\int_{B_r} |\psi|^{\frac{n}{2}} dV_g = o(r^2) \quad \text{as } r \rightarrow +\infty$$

with $A \in \mathbb{R}$, $A + \frac{n}{2} - 1 > 0$ and $q(x) \in C^0(M)$. If ψ is not identically zero, then

$$\|q_+(x)\|_{L^{\frac{n}{2}}(M)} \geq \frac{16C(n)(A + n/2 - 1)}{n^2}.$$

3 Rigidity of Kähler-Einstein manifolds

In this section, we consider complete Kähler manifolds with constant scalar curvature. Some L^p and L^∞ pinching results will be established to characterize Kähler-Einstein manifolds among complete Kähler manifolds with constant scalar curvature.

Suppose (M, g, J) is a Kähler manifold with constant scalar curvature. First, we intend to derive the Weitzenböck formula for the traceless Ricci tensor E . Note that E is a $(1, 1)$ -type Codazzi tensor and

$$|E|^2 = 2|E_{\alpha\bar{\beta}}|^2.$$

For simplicity, one may choose a normal complex coordinate system at a given point. Using (2.6), (2.7) and (2.10), a direct computation gives

$$\begin{aligned} \frac{1}{2} \Delta |E|^2 &= \nabla_\lambda \nabla_{\bar{\lambda}} |E_{\alpha\bar{\beta}}|^2 + \nabla_{\bar{\lambda}} \nabla_\lambda |E_{\alpha\bar{\beta}}|^2 \\ &= 4E_{\alpha\bar{\beta},\lambda} E_{\bar{\alpha}\beta,\bar{\lambda}} + 2E_{\alpha\bar{\beta}} E_{\bar{\alpha}\beta,\lambda\bar{\lambda}} + 2E_{\alpha\bar{\beta}} E_{\bar{\alpha}\beta,\bar{\lambda}\lambda} \\ &= 4E_{\alpha\bar{\beta},\lambda} E_{\bar{\alpha}\beta,\bar{\lambda}} + 4E_{\alpha\bar{\beta}} R_{\beta\bar{\gamma}} E_{\gamma\bar{\alpha}} + 4E_{\alpha\bar{\beta}} E_{\lambda\bar{\gamma}} R_{\gamma\bar{\alpha}\beta\bar{\lambda}} \\ &= 4E_{\alpha\bar{\beta},\lambda} E_{\bar{\alpha}\beta,\bar{\lambda}} + \frac{4m}{m+2} \text{tr}(E_{\alpha\bar{\beta}})^3 + 4E_{\alpha\bar{\beta}} E_{\lambda\bar{\gamma}} B_{\gamma\bar{\alpha}\beta\bar{\lambda}} + \frac{4R}{m+1} |E_{\alpha\bar{\beta}}|^2 \\ &= |\nabla E|^2 + \frac{4m}{m+2} \text{tr}(E_{\alpha\bar{\beta}})^3 + 4E_{\alpha\bar{\beta}} E_{\lambda\bar{\gamma}} B_{\gamma\bar{\alpha}\beta\bar{\lambda}} + \frac{2R}{m+1} |E|^2 \end{aligned} \quad (3.1)$$

where $|\nabla E|^2 = 4E_{\alpha\bar{\beta},\lambda} E_{\bar{\alpha}\beta,\bar{\lambda}}$.

In [HV], the authors deduced the Kato's inequality for a traceless Codazzi tensor. Using a similar method, we may derive the following Kato's inequality for a $(1, 1)$ -type Codazzi tensor.

Lemma 3.1. *Let C be a traceless $(1, 1)$ -type Codazzi tensor field on (M^m, g, J) . Then*

$$|\nabla |C||^2 \leq \frac{m}{m+1} |\nabla C|^2 \quad (3.2)$$

at any point where $|C| \neq 0$. In addition, the constant on the right hand side of the inequality is optimal.

Proof. Clearly the inequality (3.2) is equivalent to

$$\frac{1}{4}|\nabla|C|^2|^2 \leq \frac{m}{m+1}|C|^2|\nabla C|^2. \quad (3.3)$$

For any given point $p \in M$, one may choose a system of complex coordinates (z_α) such that

$$g_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}} \quad \text{and} \quad C_{\alpha\bar{\beta}} = \lambda_\alpha \delta_{\alpha\bar{\beta}}$$

at this point. Since $(C_{\alpha\bar{\beta}})$ is Hermitian symmetric, each eigenvalue λ_α is real. Write

$$C = C_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta + C_{\bar{\alpha}\beta} d\bar{z}^\alpha \otimes dz^\beta.$$

So

$$|C|^2 = 2 C_{\alpha\bar{\beta}} C^{\alpha\bar{\beta}} = 2 C_{\alpha\bar{\beta}} C_{\bar{\alpha}\beta}.$$

First, we compute

$$\begin{aligned} |\nabla|C|^2|^2 &= (|C|^2)_\gamma (|C|^2)_{\bar{\gamma}} + (|C|^2)_{\bar{\gamma}} (|C|^2)_\gamma \\ &= 32 C_{\alpha\bar{\beta},\gamma} C_{\bar{\alpha}\beta} C_{\mu\bar{\nu},\bar{\gamma}} C_{\bar{\mu}\nu} \\ &= 32 \left(\sum_\alpha C_{\alpha\bar{\alpha},\gamma} C_{\bar{\alpha}\alpha} \right) \left(\sum_\mu C_{\mu\bar{\mu},\bar{\gamma}} C_{\bar{\mu}\mu} \right) \\ &= 32 \sum_\gamma \left| \sum_\alpha C_{\alpha\bar{\alpha},\gamma} C_{\bar{\alpha}\alpha} \right|^2 \\ &\leq 32 \sum_\gamma \left(\sum_\alpha |C_{\alpha\bar{\alpha}}| |C_{\alpha\bar{\alpha},\gamma}| \right)^2. \end{aligned}$$

Next, since C is a $(1,1)$ -type Codazzi tensor, we discover

$$\begin{aligned} |\nabla C|^2 &= 4 \sum_{\alpha,\beta,\gamma} |C_{\alpha\bar{\beta},\gamma}|^2 \\ &= 4 \sum_\gamma \{ |C_{\gamma\bar{\gamma},\gamma}|^2 + \sum_{\alpha \neq \gamma} |C_{\alpha\bar{\alpha},\gamma}|^2 + \sum_{\alpha \neq \gamma} |C_{\alpha\bar{\gamma},\gamma}|^2 \} + \text{positive terms} \\ &\geq 4 \sum_\gamma \{ |C_{\gamma\bar{\gamma},\gamma}|^2 + 2 \sum_{\alpha \neq \gamma} |C_{\alpha\bar{\alpha},\gamma}|^2 \}. \end{aligned}$$

In order to prove (3.3), one only needs to verify the following inequality

$$\sum_\gamma \left(\sum_\alpha |C_{\alpha\bar{\alpha}}| |C_{\alpha\bar{\alpha},\gamma}| \right)^2 \leq \frac{m}{2(m+1)} |C|^2 \sum_\gamma \{ |C_{\gamma\bar{\gamma},\gamma}|^2 + 2 \sum_{\alpha \neq \gamma} |C_{\alpha\bar{\alpha},\gamma}|^2 \}. \quad (3.4)$$

Set $\mu_\alpha = C_{\alpha\bar{\alpha},\gamma}$ for any fixed γ . Note that $\lambda_\alpha = C_{\alpha\bar{\alpha}}$. Consequently

$$\sum_\alpha \lambda_\alpha = 0, \quad \sum_\alpha \mu_\alpha = 0, \quad (3.5)$$

since C is traceless. It follows from (3.5) and the Cauchy-Schwarz inequality that

$$|\mu_\gamma|^2 = \left| \sum_{\alpha \neq \gamma} \mu_\alpha \right|^2 \leq \left(\sum_{\alpha \neq \gamma} |\mu_\alpha| \right)^2 \leq (m-1) \left(\sum_{\alpha \neq \gamma} |\mu_\alpha|^2 \right).$$

Hence

$$\begin{aligned}
|\mu_\gamma|^2 + 2 \sum_{\alpha \neq \gamma} |\mu_\alpha|^2 &= |\mu_\gamma|^2 + \frac{1}{m} [(m-1) \sum_{\alpha \neq \gamma} |\mu_\alpha|^2 + (m+1) \sum_{\alpha \neq \gamma} |\mu_\alpha|^2] \\
&\geq |\mu_\gamma|^2 + \frac{1}{m} [|\mu_\gamma|^2 + (m+1) \sum_{\alpha \neq \gamma} |\mu_\alpha|^2] \\
&= \frac{m+1}{m} \sum_{\alpha} |\mu_\alpha|^2.
\end{aligned} \tag{3.6}$$

Using (3.6) and the Cauchy-Schwarz inequality, we find

$$\begin{aligned}
\frac{(\sum_{\alpha} |\lambda_{\alpha}|^2)(|\mu_\gamma|^2 + 2 \sum_{\alpha \neq \gamma} |\mu_\alpha|^2)}{(\sum_{\alpha} |\lambda_{\alpha}| |\mu_\alpha|)^2} &\geq \frac{m+1}{m} \frac{(\sum_{\alpha} |\lambda_{\alpha}|^2)(\sum_{\alpha} |\mu_\alpha|^2)}{(\sum_{\alpha} |\lambda_{\alpha}| |\mu_\alpha|)^2} \\
&\geq \frac{m+1}{m}
\end{aligned}$$

which implies immediately the inequality (3.4). ■

Another equivalent expression of (2.5) is

$$\begin{aligned}
R_{\alpha\bar{\beta}\gamma\bar{\delta}} &= B_{\alpha\bar{\beta}\gamma\bar{\delta}} - \frac{1}{m+2} (E_{\alpha\bar{\beta}} g_{\gamma\bar{\delta}} + E_{\gamma\bar{\beta}} g_{\alpha\bar{\delta}} + g_{\alpha\bar{\beta}} E_{\gamma\bar{\delta}} + g_{\gamma\bar{\beta}} E_{\alpha\bar{\delta}}) \\
&\quad - \frac{R}{m(m+1)} (g_{\alpha\bar{\beta}} g_{\gamma\bar{\delta}} + g_{\gamma\bar{\beta}} g_{\alpha\bar{\delta}}),
\end{aligned}$$

which tells us that the curvature tensor of a Kähler manifold can be decomposed into three orthogonal parts with respect to the Hermitian structure. Now we want to estimate the third term on the right hand side of (3.1) by using the same technique as in [Hu] for treating a similar contracted term of the traceless Ricci tensor and the Weyl tensor.

Lemma 3.2. *The inequality*

$$|E_{\alpha\bar{\beta}} E_{\lambda\bar{\gamma}} B_{\gamma\bar{\alpha}\beta\bar{\lambda}}| \leq \frac{1}{4} \sqrt{\frac{2m^2 + 4m + 3}{2(m+1)(m+2)}} |B| |E|^2$$

holds on any Kähler m -manifold.

Proof. We define a curvature-like tensor

$$\begin{aligned}
V &= (E_{\alpha\bar{\beta}} E_{\delta\bar{\gamma}} + E_{\alpha\bar{\gamma}} E_{\delta\bar{\beta}}) dz^\alpha \otimes dz^{\bar{\beta}} \otimes dz^{\bar{\gamma}} \otimes dz^\delta \\
&\quad + (E_{\beta\bar{\alpha}} E_{\gamma\bar{\delta}} + E_{\gamma\bar{\alpha}} E_{\beta\bar{\delta}}) dz^{\bar{\alpha}} \otimes dz^\beta \otimes dz^{\bar{\gamma}} \otimes dz^{\bar{\delta}} \\
&\quad - (E_{\beta\bar{\alpha}} E_{\delta\bar{\gamma}} + E_{\delta\bar{\alpha}} E_{\beta\bar{\gamma}}) dz^{\bar{\alpha}} \otimes dz^\beta \otimes dz^{\bar{\gamma}} \otimes dz^\delta \\
&\quad - (E_{\alpha\bar{\beta}} E_{\gamma\bar{\delta}} + E_{\alpha\bar{\delta}} E_{\gamma\bar{\beta}}) dz^\alpha \otimes dz^{\bar{\beta}} \otimes dz^\gamma \otimes dz^{\bar{\delta}}.
\end{aligned} \tag{3.7}$$

Clearly V has the same symmetries as the curvature tensor of a Kähler manifold. So it can be decomposed into three orthogonal parts with respect to the Hermitian

structure: $V = V_1 + V_2 + V_3$. Here V_1 , V_2 and V_3 correspond to the ‘Bochner curvature’ part, the ‘traceless Ricci’ part and the ‘scalar curvature’ part of V respectively. To express V_i explicitly, let’s introduce

$$V_{\alpha\bar{\beta}}^E = V_{\alpha\bar{\beta}}^{Ric} - \frac{K}{m} g_{\alpha\bar{\beta}}$$

where

$$V_{\alpha\bar{\beta}}^{Ric} = g^{\gamma\bar{\delta}} V_{\alpha\bar{\beta}\delta\gamma} = g^{\gamma\bar{\delta}} E_{\alpha\bar{\delta}} E_{\gamma\bar{\beta}}$$

and

$$K = g^{\alpha\bar{\beta}} V_{\alpha\bar{\beta}}^{Ric} = \frac{1}{2} |E|^2.$$

Therefore the components of V_2 and V_3 are given by

$$(V_2)_{\alpha\bar{\beta}\gamma\bar{\delta}} = -\frac{1}{m+2} (V_{\alpha\bar{\beta}}^E g_{\gamma\bar{\delta}} + V_{\gamma\bar{\beta}}^E g_{\alpha\bar{\delta}} + g_{\alpha\bar{\beta}} V_{\gamma\bar{\delta}}^E + g_{\gamma\bar{\beta}} V_{\alpha\bar{\delta}}^E)$$

and

$$(V_3)_{\alpha\bar{\beta}\gamma\bar{\delta}} = -\frac{K}{m(m+1)} (g_{\alpha\bar{\beta}} g_{\gamma\bar{\delta}} + g_{\gamma\bar{\beta}} g_{\alpha\bar{\delta}}).$$

As before, we may assume $g_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}}$ at a given point. From (2.4) and (3.7), we have

$$8E_{\alpha\bar{\beta}} E_{\lambda\bar{\gamma}} B_{\gamma\bar{\alpha}\beta\bar{\lambda}} = \langle B, V \rangle = \langle B, V_1 \rangle \quad (3.8)$$

where $\langle \cdot, \cdot \rangle$ denotes the Hermitian inner product induced from g . Set

$$Z = E_{\alpha\bar{\beta}} E_{\beta\bar{\gamma}} E_{\gamma\bar{\delta}} E_{\delta\bar{\alpha}}.$$

A direct calculation yields that

$$\begin{aligned} \frac{1}{4} |V|^2 &= (E_{\alpha\bar{\beta}} E_{\gamma\bar{\delta}} + E_{\alpha\bar{\delta}} E_{\gamma\bar{\beta}}) (E_{\bar{\alpha}\beta} E_{\bar{\gamma}\delta} + E_{\bar{\alpha}\delta} E_{\bar{\gamma}\beta}) \\ &= \frac{1}{2} |E|^4 + 2Z, \end{aligned} \quad (3.9)$$

$$\begin{aligned} \frac{1}{4} |V_2|^2 &= \frac{4}{m+2} V_{\alpha\bar{\beta}}^E V_{\bar{\alpha}\beta}^E \\ &= \frac{4}{m+2} (E_{\alpha\bar{\lambda}} E_{\lambda\bar{\beta}} - \frac{1}{2m} |E|^2 \delta_{\alpha\bar{\beta}}) (E_{\bar{\alpha}\mu} E_{\bar{\mu}\beta} - \frac{1}{2m} |E|^2 \delta_{\bar{\alpha}\beta}) \\ &= \frac{4}{m+2} Z - \frac{1}{m(m+2)} |E|^4, \end{aligned} \quad (3.10)$$

and

$$\frac{1}{4} |V_3|^2 = \frac{1}{2m(m+1)} |E|^4. \quad (3.11)$$

From (3.9), (3.10) and (3.11), we deduce

$$\begin{aligned}
|V_1|^2 &= |V|^2 - |V_2|^2 - |V_3|^2 \\
&= \frac{8m}{m+2}Z + \frac{2(m+1)(m+2)+2}{(m+1)(m+2)}|E|^4 \\
&\leq \frac{2m}{m+2}|E|^4 + \frac{2(m+1)(m+2)+2}{(m+1)(m+2)}|E|^4 \\
&= \frac{4m^2+8m+6}{(m+1)(m+2)}|E|^4.
\end{aligned} \tag{3.12}$$

It follows from (3.8) and (3.12) that

$$\begin{aligned}
|E_{\alpha\bar{\beta}}E_{\lambda\bar{\gamma}}B_{\gamma\bar{\alpha}\beta\bar{\lambda}}| &\leq \frac{1}{8}|\langle B, V_1 \rangle| \\
&\leq \frac{1}{8}|B||V_1| \\
&\leq \frac{1}{4}\sqrt{\frac{2m^2+4m+3}{2(m+1)(m+2)}}|B||E|^2.
\end{aligned}$$

■

Using Lemmas 2.1 and 3.2, we get from (3.1) that

$$\frac{1}{2}\Delta|E|^2 \geq |\nabla E|^2 - \frac{(m-2)\sqrt{2m}}{(m+2)\sqrt{(m-1)}}|E|^3 - \sqrt{\frac{2m^2+4m+3}{2(m+1)(m+2)}}|B||E|^2 + \frac{2R}{m+1}|E|^2 \tag{3.13}$$

and thus using Lemma 3.1, we find

$$\begin{aligned}
|E|\Delta|E| &\geq |\nabla E|^2 - |\nabla|E||^2 - \frac{(m-2)\sqrt{2m}}{(m+2)\sqrt{(m-1)}}|E|^3 - \sqrt{\frac{2m^2+4m+3}{2(m+1)(m+2)}}|B||E|^2 \\
&\quad + \frac{2R}{m+1}|E|^2 \\
&\geq \frac{1}{m}|\nabla|E||^2 + \frac{2R}{m+1}|E|^2 - \frac{(m-2)\sqrt{2m}}{(m+2)\sqrt{(m-1)}}|E|^3 \\
&\quad - \sqrt{\frac{2m^2+4m+3}{2(m+1)(m+2)}}|B||E|^2.
\end{aligned} \tag{3.14}$$

Theorem 3.1. *Let (M, g, J) be a complete noncompact Kähler m -manifold ($m \geq 2$) with zero scalar curvature and positive Yamabe constant $\Lambda(M, g)$. Assume that*

$$\sqrt{2}\|E\|_{L^m(M)} + \|B\|_{L^m(M)} < \frac{4\Lambda(M, g)(m^2 - m + 1)}{m^3} \sqrt{\frac{2(m+1)(m+2)}{2m^2+4m+3}}. \tag{3.15}$$

Then M is a Ricci-flat Kähler manifold.

Proof. Since $R = 0$, the differential inequality (3.14) becomes

$$|E|\Delta|E| + \left(\frac{m-2}{m+2} \sqrt{\frac{2m}{m-1}} |E| + \sqrt{\frac{2m^2+4m+3}{2(m+1)(m+2)}} |B| \right) |E|^2 \geq \frac{1}{m} |\nabla|E||^2. \quad (3.16)$$

Under the assumptions that $\Lambda(M, g) > 0$ and $R = 0$, the following Euclidean-type Sobolev inequality

$$\Lambda(M, g) \left(\int_M |u|^{\frac{2m}{m-1}} dV_g \right)^{\frac{m-1}{m}} \leq \int_M |\nabla u|^2 dV_g$$

holds for any $u \in C_0^\infty(M)$.

We have to show that $|E| = 0$. Clearly (3.15) implies that $\int_M |E|^m dV_g$ is finite, and thus

$$\int_{B_r} |E|^m dV_g = o(r^2) \quad \text{as } r \rightarrow \infty.$$

If $|E|$ is not identically zero, applying Lemma 2.2 to (3.16), we get

$$\left\| \frac{m-2}{m+2} \sqrt{\frac{2m}{m-1}} |E| + \sqrt{\frac{2m^2+4m+3}{2(m+1)(m+2)}} |B| \right\|_{L^m(M)} \geq \frac{4\Lambda(M, g)(m^2 - m + 1)}{m^3}.$$

Note that $\sqrt{\frac{2m^2+4m+3}{2(m+1)(m+2)}} > \frac{m-2}{m+2} \sqrt{\frac{m}{m-1}}$ for $m \geq 2$. Consequently

$$\sqrt{\frac{2m^2+4m+3}{2(m+1)(m+2)}} (\sqrt{2} \|E\|_{L^m(M)} + \|B\|_{L^m(M)}) \geq \frac{4\Lambda(M, g)(m^2 - m + 1)}{m^3}$$

which contradicts to (3.15). Hence we conclude that $E = 0$, that is, (M, g, J) is Kähler-Einstein. ■

Next we deal with the case that $R < 0$. Although in this case, the Sobolev inequality (2.2) implies the Euclidean-type Sobolev inequality (2.10) with $C(n) = \Lambda(M, g)$ and $n = 2m$, the direct application of Lemma 2.2 to (3.16) does not yield a nice gap result as in Theorem 3.1. Inspired by a technique in [Ki], we establish the following result.

Theorem 3.2. *Let (M, g, J) be a complete noncompact Kähler m -manifold ($m \geq 3$) with constant negative scalar curvature R and positive Yamabe constant $\Lambda(M, g)$. Suppose that*

$$\int_{B_r} |E|^2 dV_g = o(r^2) \quad \text{as } r \rightarrow \infty$$

where B_r denotes a geodesic ball of radius r relative to some fixed point $x_0 \in M$. If

$$\sqrt{2} \|E\|_{L^m(M)} + \|B\|_{L^m(M)} < \frac{\Lambda(M, g)(m+1)}{m} \sqrt{\frac{2(m+1)(m+2)}{2m^2+4m+3}}, \quad (3.17)$$

then (M, g, J) is Kähler-Einstein.

Proof. Set $u = |E|$. For any test function $0 \leq \phi \in C_0^\infty(M)$, we get from (3.14) that

$$\begin{aligned} \int_M u(\Delta u)\phi^2 dV_g &\geq \int_M \left\{ \frac{1}{m} |\nabla u|^2 \phi^2 + \frac{2R}{m+1} u^2 \phi^2 - \frac{m-2}{m+2} \sqrt{\frac{2m}{m-1}} u^3 \phi^2 \right. \\ &\quad \left. - \sqrt{\frac{2m^2+4m+3}{2(m+1)(m+2)}} |B| u^2 \phi^2 \right\} dV_g. \end{aligned} \quad (3.18)$$

Using integration by parts and the Schwarz inequality, we deduce

$$\begin{aligned} \int_M u(\Delta u)\phi^2 dV &= - \int_M |\nabla u|^2 \phi^2 dV_g - 2 \int_M \phi u \langle \nabla u, \nabla \phi \rangle dV_g \\ &\leq (\varepsilon_1 - 1) \int_M |\nabla u|^2 \phi^2 dV_g + \varepsilon_1^{-1} \int_M |\nabla \phi|^2 u^2 dV_g \end{aligned} \quad (3.19)$$

for any $\varepsilon_1 > 0$. It follows from (3.18) and (3.19) that

$$\begin{aligned} (1 + \frac{1}{m} - \varepsilon_1) \int_M |\nabla u|^2 \phi^2 dV_g &\leq \int_M \left\{ \varepsilon_1^{-1} |\nabla \phi|^2 u^2 + \phi^2 \left[\frac{m-2}{m+2} \sqrt{\frac{2m}{m-1}} u^3 \right. \right. \\ &\quad \left. \left. + \sqrt{\frac{2m^2+4m+3}{2(m+1)(m+2)}} |B| u^2 - \frac{2R}{m+1} u^2 \right] \right\} dV_g. \end{aligned} \quad (3.20)$$

From the Sobolev inequality (2.2) and the Schwarz inequality, we find

$$\begin{aligned} \Lambda(M, g) \left(\int_M (\phi u)^{\frac{2m}{m-1}} dV_g \right)^{\frac{m-1}{m}} &\leq \int_M \left\{ (1 + \varepsilon_2) |\nabla u|^2 \phi^2 + (1 + \varepsilon_2^{-1}) |\nabla \phi|^2 u^2 \right. \\ &\quad \left. + \frac{m-1}{2m-1} R(\phi u)^2 \right\} dV_g \end{aligned} \quad (3.21)$$

for any $\varepsilon_2 > 0$. Then (3.20) and (3.21) imply

$$\begin{aligned} \Lambda(M, g) \left(\int_M (\phi u)^{\frac{2m}{m-1}} dV_g \right)^{\frac{m-1}{m}} &\leq \int_M \left\{ A_1 |\nabla \phi|^2 u^2 + A_2 R \phi^2 u^2 \right. \\ &\quad \left. + A_3 |B| \phi^2 u^2 + A_4 \phi^2 u^3 \right\} dV_g \end{aligned} \quad (3.22)$$

where

$$\begin{aligned} A_1 &= \frac{1 + \varepsilon_2}{(1 + m^{-1} - \varepsilon_1) \varepsilon_1} + 1 + \varepsilon_2^{-1}, \\ A_2 &= \frac{m-1}{2m-1} - \frac{2(1 + \varepsilon_2)}{(m+1)(1 + m^{-1} - \varepsilon_1)}, \\ A_3 &= \frac{1 + \varepsilon_2}{1 + m^{-1} - \varepsilon_1} \sqrt{\frac{2m^2+4m+3}{2(m+1)(m+2)}}, \\ A_4 &= \frac{(1 + \varepsilon_2)(m-2)}{(1 + m^{-1} - \varepsilon_1)(m+2)} \sqrt{\frac{2m}{m-1}}. \end{aligned}$$

Note that $A_2 > 0$ for $m \geq 3$ and sufficiently small ε_1 and ε_2 . Under the assumption (3.17), we may choose sufficiently small ε_1 and ε_2 such that

$$\sqrt{2}\|E\|_{L^m(M)} + \|B\|_{L^m(M)} < \frac{\Lambda(M, g)(1 + m^{-1} - 2\varepsilon_1)}{1 + 2\varepsilon_2} \sqrt{\frac{2(m+1)(m+2)}{2m^2 + 4m + 3}}. \quad (3.23)$$

Moreover, the sufficiently small ε_1 and ε_2 also ensure

$$\left\{ \frac{m-1}{2m-1} - \frac{2(1+\varepsilon_2)}{(m+1)(1+m^{-1}-\varepsilon_1)} \right\} R\phi^2 u^2 \leq 0. \quad (3.24)$$

Since $\sqrt{2}A_3 \geq A_4$ for $m \geq 3$, we get from (3.22) and (3.24) that

$$\Lambda(M, g) \left(\int_M (\phi u)^{\frac{2m}{m-1}} dV_g \right)^{\frac{m-1}{m}} \leq \int_M \{A_1 |\nabla \phi|^2 u^2 + A_3 [|B| + \sqrt{2}u] \phi^2 u^2\} dV_g. \quad (3.25)$$

The Hölder inequality gives

$$\begin{aligned} \int_M |B| \phi^2 u^2 dV_g &\leq \left(\int_M |B|^m dV_g \right)^{\frac{1}{m}} \left(\int_M (\phi u)^{\frac{2m}{m-1}} dV_g \right)^{\frac{m-1}{m}} \\ \int_M u^3 \phi^2 dV_g &\leq \left(\int_M |u|^m dV_g \right)^{\frac{1}{m}} \left(\int_M (\phi u)^{\frac{2m}{m-1}} dV_g \right)^{\frac{m-1}{m}}. \end{aligned} \quad (3.26)$$

Hence we may combine (3.23), (3.25) and (3.26) to find

$$\begin{aligned} &\Lambda(M, g) \left(\int_M (\phi u)^{\frac{2m}{m-1}} dV_g \right)^{\frac{m-1}{m}} \\ &\leq A_1 \int_M |\nabla \phi|^2 u^2 dV_g + \frac{\Lambda(M, g)(1 + \varepsilon_2)(1 + m^{-1} - 2\varepsilon_1)}{(1 + 2\varepsilon_2)((1 + m^{-1} - \varepsilon_1))} \left(\int_M (\phi u)^{\frac{2m}{m-1}} dV_g \right)^{\frac{m-1}{m}}. \end{aligned}$$

Consequently

$$\Lambda(M, g) \left[1 - \frac{(1 + \varepsilon_2)(1 + m^{-1} - 2\varepsilon_1)}{(1 + 2\varepsilon_2)((1 + m^{-1} - \varepsilon_1))} \right] \left(\int_M (\phi u)^{\frac{2m}{m-1}} dV_g \right)^{\frac{m-1}{m}} \leq A_1 \int_M |\nabla \phi|^2 u^2 dV_g. \quad (3.27)$$

Now we let $\phi = \phi_r$ be a family of cut-off functions satisfying

$$\phi_r \equiv 1 \text{ on } B_r; \quad \phi_r \equiv 0 \text{ off } B_{2r}; \quad |\nabla \phi_r| \leq \frac{2}{r} \text{ on } B_{2r} - B_r.$$

Then (3.27) becomes

$$\Lambda(M, g) \left[1 - \frac{(1 + \varepsilon_2)(1 + m^{-1} - 2\varepsilon_1)}{(1 + 2\varepsilon_2)((1 + m^{-1} - \varepsilon_1))} \right] \left(\int_{B_r} u^{\frac{2m}{m-1}} dV_g \right)^{\frac{m-1}{m}} \leq \frac{4A_1}{r^2} \int_{B_{2r}} u^2 dV_g. \quad (3.28)$$

Letting $r \rightarrow \infty$ in (3.28), we get

$$\int_M u^{\frac{2m}{m-1}} dV_g = 0,$$

that is, $u \equiv 0$. Hence (M, g, J) is Kähler-Einstein. ■

Now we consider the case that $R > 0$. The Bonnet-Myers theorem in Riemannian geometry implies that any complete Einstein manifold with positive scalar curvature must be compact. Following in this section are two rigidity results about compact Kähler manifolds with positive scalar curvature. Recall that if (M, g) is compact, the positivity of the scalar curvature guarantees the positivity of the Yamabe constant $\Lambda(M, g)$.

Theorem 3.3. *Let (M, g, J) be a compact Kähler m -manifold ($m \geq 2$) with constant positive scalar curvature R . If*

$$\sqrt{2}\|E\|_{L^m(M)} + \|B\|_{L^m(M)} < \Lambda(M, g)P(m)\sqrt{\frac{2(m+1)(m+2)}{2m^2+4m+3}} \quad (3.29)$$

where $P(2) = 3/2$ and $P(m) = \frac{2(2m-1)}{m^2-1}$ for $m \geq 3$, then (M, g, J) is Kähler-Einstein.

Proof. By integrating (3.14) and using the Hölder inequality, we have

$$\begin{aligned} \int_M |\nabla|E||^2 dV_g &\leq \frac{1}{1+m^{-1}} \|q\|_{L^m(M)} \left(\int_M |E|^{\frac{2m}{m-1}} dV_g \right)^{\frac{m-1}{m}} \\ &\quad - \frac{2R}{(m+1)(1+m^{-1})} \int_M |E|^2 dV_g \end{aligned} \quad (3.30)$$

where

$$q(x) = \frac{m-2}{m+2} \sqrt{\frac{2m}{m-1}} |E| + \sqrt{\frac{2m^2+4m+3}{2(m+1)(m+2)}} |B|.$$

Since the condition $R > 0$ implies $\Lambda(M, g) > 0$, (2.2) gives

$$\Lambda(M, g) \left(\int_M |E|^{\frac{2m}{m-1}} dV_g \right)^{\frac{m-1}{m}} \leq \int_M |\nabla|E||^2 dV_g + \frac{(m-1)R}{2m-1} \int_M |E|^2 dV_g. \quad (3.31)$$

Substituting (3.31) into (3.30) leads to

$$\begin{aligned} \int_M |\nabla|E||^2 dV_g &\leq \frac{1}{\Lambda(M, g)(1+m^{-1})} \|q\|_{L^m(M)} \int_M [|\nabla|E||^2 + \frac{(m-1)R}{2m-1} |E|^2] dV_g \\ &\quad - \frac{2R}{(m+1)(1+m^{-1})} \int_M |E|^2 dV_g. \end{aligned}$$

Consequently

$$\begin{aligned} &\left(1 - \frac{1}{\Lambda(M, g)(1+m^{-1})} \|q\|_{L^m(M)} \right) \int_M |\nabla|E||^2 dV_g + \\ &\left(\frac{2}{(m+1)(1+m^{-1})} - \frac{(m-1)}{\Lambda(M, g)(1+m^{-1})(2m-1)} \|q\|_{L^m(M)} \right) R \int_M |E|^2 dV_g \leq 0. \end{aligned} \quad (3.32)$$

Note again that

$$\|q\|_{L^m(M)} \leq \sqrt{\frac{2m^2 + 4m + 3}{2(m+1)(m+2)}} (\sqrt{2}|E|_{L^m(M)} + \|B\|_{L^m(M)})$$

for $m \geq 2$. Clearly the condition (3.29) guarantees that the two terms on the left hand side of the inequality (3.32) are nonnegative. Thus we may conclude that $E = 0$. ■

Theorem 3.4. *Let (M, g, J) be a compact Kähler m -manifold ($m \geq 2$) with constant positive scalar curvature R . If*

$$\sqrt{2}|E| + |B| \leq \frac{2}{m+1} \sqrt{\frac{2(m+1)(m+2)}{2m^2 + 4m + 3}} R,$$

then (M, g, J) is Kähler-Einstein.

Proof. From (3.13), we have

$$\begin{aligned} \frac{1}{2} \Delta |E|^2 &\geq \left\{ \frac{2R}{m+1} - \frac{m-2}{m+2} \sqrt{\frac{2m}{m-1}} |E| - \sqrt{\frac{2m^2 + 4m + 3}{2(m+1)(m+2)}} |B| \right\} |E|^2 \\ &= \left\{ \frac{2R}{m+1} - \sqrt{\frac{2m^2 + 4m + 3}{2(m+1)(m+2)}} (\sqrt{2}|E| + |B|) \right\} |E|^2 \\ &\quad + \sqrt{2} \left(\sqrt{\frac{2m^2 + 4m + 3}{2(m+1)(m+2)}} - \frac{m-2}{m+2} \sqrt{\frac{m}{m-1}} \right) |E|^3. \end{aligned} \quad (3.33)$$

Since $\sqrt{\frac{2m^2 + 4m + 3}{2(m+1)(m+2)}} > \frac{m-2}{m+2} \sqrt{\frac{m}{m-1}}$, the integration of (3.33) leads to $E = 0$, that is, (M, g, J) is Kähler-Einstein. ■

4 Rigidity of complex space forms

In this section, we establish some rigidity results characterizing complex space forms among complete Kähler-Einstein manifolds and complete Kähler manifolds with constant scalar curvature respectively.

Suppose (M, g, J) is a Kähler-Einstein manifold of dimension m ($m \geq 2$). The Einsteinian condition implies directly that the scalar curvature R is constant and (2.6) becomes

$$B_{\alpha\bar{\beta}\gamma\bar{\delta}} = R_{\alpha\bar{\beta}\gamma\bar{\delta}} + \frac{R}{m(m+1)} (g_{\alpha\bar{\beta}} g_{\gamma\bar{\delta}} + g_{\gamma\bar{\beta}} g_{\alpha\bar{\delta}}). \quad (4.1)$$

In this circumstance the Bochner tensor measures the deviation of a Kähler-Einstein metric from the metric with constant holomorphic sectional curvature.

We want next to derive the Weitzenböck formula for the Bochner tensor B . Note that B is regarded as a real tensor in $\Lambda^{1,1}(M) \otimes \Lambda^{1,1}(M)$. As in §3, we take a normal complex coordinate system at a given point. So

$$|B|^2 = 4|B_{\alpha\bar{\beta}\gamma\bar{\delta}}|^2.$$

Using (2.4), (2.7), (2.10) and (4.1), a direct computation gives

$$\begin{aligned} \frac{1}{2}\Delta|B|^2 &= 2(\nabla_\lambda \nabla_{\bar{\lambda}} |B_{\alpha\bar{\beta}\gamma\bar{\delta}}|^2 + \nabla_{\bar{\lambda}} \nabla_\lambda |B_{\alpha\bar{\beta}\gamma\bar{\delta}}|^2) \\ &= 8B_{\alpha\bar{\beta}\gamma\bar{\delta},\lambda} B_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta},\bar{\lambda}} + 4B_{\alpha\bar{\beta}\gamma\bar{\delta},\lambda\bar{\lambda}} B_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} + 4B_{\alpha\bar{\beta}\gamma\bar{\delta},\bar{\lambda}\lambda} B_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} \\ &= 8B_{\alpha\bar{\beta}\gamma\bar{\delta},\lambda} B_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta},\bar{\lambda}} + 4B_{\alpha\bar{\beta}\lambda\bar{\delta},\gamma\bar{\lambda}} B_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} + 4B_{\alpha\bar{\beta}\gamma\bar{\lambda},\bar{\delta}\lambda} B_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} \\ &= 8|B_{\alpha\bar{\beta}\gamma\bar{\delta},\lambda}|^2 + 8B_{\bar{\beta}v\lambda\bar{\delta}} B_{\bar{v}\alpha\gamma\bar{\lambda}} B_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} - 16B_{\alpha\bar{\delta}\bar{v}\lambda} B_{v\bar{\lambda}\bar{\beta}\gamma} B_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} + \frac{8R}{m}|B_{\alpha\bar{\beta}\gamma\bar{\delta}}|^2. \end{aligned} \quad (4.2)$$

Let us introduce two $m^2 \times m^2$ Hermitian matrices H and K as follows:

$$\begin{aligned} H &= (H_{ab\bar{c}\bar{d}}) = (B_{\bar{c}ab\bar{d}}), \\ K &= (K_{a\bar{b}c\bar{d}}) = (B_{a\bar{b}\bar{c}d}). \end{aligned}$$

Then (4.2) becomes

$$\frac{1}{2}\Delta|B|^2 = |\nabla B|^2 + 8\text{tr}(H^3) - 16\text{tr}(K^3) + \frac{2R}{m}|B|^2 \quad (4.3)$$

where $|\nabla B|^2 = 8|B_{\alpha\bar{\beta}\gamma\bar{\delta},\lambda}|^2$. In view of (2.4) and (2.5), we see

$$\begin{aligned} \text{tr}(H) &= \text{tr}(K) = 0, \\ \text{tr}(H^2) &= \text{tr}(K^2) = \frac{1}{4}|B|^2. \end{aligned}$$

Consequently Lemma 2.1 yields

$$|\text{tr}(H^3)| \leq \frac{m^2 - 2}{8\sqrt{m^2(m^2 - 1)}}|B|^3, \quad (4.4)$$

and

$$|\text{tr}(K^3)| \leq \frac{m^2 - 2}{8\sqrt{m^2(m^2 - 1)}}|B|^3. \quad (4.5)$$

From (4.3), (4.4) and (4.5), we deduce

$$\frac{1}{2}\Delta|B|^2 \geq |\nabla B|^2 - \frac{3(m^2 - 2)}{\sqrt{m^2(m^2 - 1)}}|B|^3 + \frac{2R}{m}|B|^2. \quad (4.6)$$

In order to estimate the first term on the right hand side of (4.6), we need the following

Lemma 4.1. ([BKN]) *Let T_1 and T_2 be tensors having the same symmetries as the curvature tensor and the covariant derivative of the curvature tensor of an Einstein metric on n -manifold respectively. Then there exists $\delta(n)$ such that*

$$(1 + \delta(n))|\langle T_1, T_2 \rangle|^2 \leq |T_1|^2 |T_2|^2,$$

where $\langle T_1, T_2 \rangle$ is a 1-form defined by $\langle T_1, T_2 \rangle(X) = \langle T_1, T_2(X) \rangle$ for a tangent X . Moreover, if g is Kähler, we can take $\delta(n) = \frac{4}{n+2} = \frac{2}{m+1}$, where $n = 2m$.

By applying Lemma 4.1 to $T_1 = B$ and $T_2 = \nabla B$, we find

$$\frac{1}{4}|\nabla|B|^2|^2 = |\langle B, \nabla B \rangle|^2 \leq \frac{m+1}{m+3}|B|^2|\nabla B|^2.$$

Note also that $|\nabla|B|^2|^2 = 4|B|^2|\nabla|B||^2$. Consequently

$$|\nabla B|^2 \geq \frac{m+3}{m+1}|\nabla|B||^2. \quad (4.7)$$

Hence (4.6) and (4.7) imply

$$|B|\triangle|B| \geq \frac{2}{m+1}|\nabla|B||^2 - \frac{3(m^2-2)}{\sqrt{m^2(m^2-1)}}|B|^3 + \frac{2R}{m}|B|^2. \quad (4.8)$$

First, we consider the case that $R = 0$. As a result of Lemma 2.2, we have

Theorem 4.1. *Let (M, g, J) be a complete noncompact Kähler-Einstein m -manifold ($m \geq 2$) with $R = 0$ and $\Lambda(M, g) > 0$. If*

$$\|B\|_{L^m(M)} < \frac{4\Lambda(M, g)(m^2+1)\sqrt{m^2-1}}{3m(m+1)(m^2-2)}, \quad (4.9)$$

then (M, g, J) is of constant holomorphic sectional curvature 0. Furthermore, if M is simply connected, then (M, g, J) is biholomorphically isometric to the complex Euclidean space C^m .

Proof. Since $R = 0$, the Sobolev inequality (2.2) provides an Euclidean-type Sobolev inequality with Sobolev constant $C(2m) = \Lambda(M, g)$, and (4.8) becomes

$$|B|\triangle|B| + \left(\frac{3(m^2-2)}{\sqrt{m^2(m^2-1)}}|B| \right) |B|^2 \geq \frac{2}{m+1}|\nabla|B||^2. \quad (4.10)$$

The assumption (4.9) implies that

$$\int_{B_r} |B|^m dV_g = o(r^2) \quad \text{as } r \rightarrow \infty.$$

Thus, if $|B|$ is not identically zero, we get from Lemma 2.2 and (4.10) that

$$\|B\|_{L^m(M)} \geq \frac{4\Lambda(M, g)(m^2+1)\sqrt{m^2-1}}{3m(m+1)(m^2-2)}$$

which contradicts to (4.9). Thus $B = 0$ and therefore (4.1) yields that $R_{\alpha\bar{\beta}\gamma\bar{\delta}} = 0$. This shows that (M, g, J) is of constant holomorphic sectional curvature 0. Consequently, if M is simply connected, then (M, g, J) is biholomorphically isometric to C^m (cf. Theorem 7.9 in [KN]). ■

Next we present the following rigidity result for the case $R < 0$. Since its proof goes almost the same way as that for Theorem 3.2, we will describe the argument briefly .

Theorem 4.2. *Let (M, g, J) be a complete noncompact Kähler-Einstein m -manifold ($m \geq 4$) with $R < 0$ and $\Lambda(M, g) > 0$. Suppose $\int_{B_r} |B|^2 dV_g = o(r^2)$ as $r \rightarrow \infty$. If*

$$\|B\|_{L^m(M)} < \frac{\Lambda(M, g)(m+3)\sqrt{m^2(m^2-1)}}{3(m+1)(m^2-2)}, \quad (4.11)$$

then (M, g, J) is of constant holomorphic sectional curvature $\frac{2R}{m(m+1)}$.

Proof. Set $v = |B|$. For any test function $\phi \in C_0^\infty(M)$, it follows from (4.8) that

$$\int_M v(\Delta v)\phi^2 dV_g \geq \int_M \left\{ \frac{2}{m+1} |\nabla v|^2 \phi^2 + \frac{2R}{m} v^2 \phi^2 - \frac{3(m^2-2)}{\sqrt{m^2(m^2-1)}} v^3 \phi^2 \right\} dV_g. \quad (4.12)$$

As we derive (3.22) from (3.18), the same process allows us to get from (4.12) the following inequality

$$\Lambda(M, g) \left(\int_M (\phi v)^{\frac{2m}{m-1}} dV_g \right)^{\frac{m-1}{m}} \leq \int_M \{B_1 |\nabla \phi|^2 v^2 + B_2 R \phi^2 v^2 + B_3 \phi^2 v^3\} dV_g,$$

where

$$\begin{aligned} B_1 &= \frac{1 + \varepsilon_2}{\varepsilon_1 [1 + 2(m+1)^{-1} - \varepsilon_1]} + 1 + \varepsilon_2^{-1}, \\ B_2 &= \frac{m-1}{2m-1} - \frac{2(1 + \varepsilon_2)}{m[1 + 2(m+1)^{-1} - \varepsilon_1]}, \\ B_3 &= \frac{3(1 + \varepsilon_2)(m^2-2)}{[1 + 2(m+1)^{-1} - \varepsilon_1] \sqrt{m^2(m^2-1)}}. \end{aligned}$$

Note that $B_2 > 0$ for $m \geq 4$ and sufficiently small ε_1 and ε_2 . Since $R < 0$, we use the Hölder inequality to find

$$\begin{aligned} \Lambda(M, g) \left(\int_M (\phi v)^{\frac{2m}{m-1}} dV_g \right)^{\frac{m-1}{m}} &\leq \int_M \{B_1 |\nabla \phi|^2 v^2 + B_3 \phi^2 v^3\} dV_g \\ &\leq B_1 \int_M |\nabla \phi|^2 v^2 dV_g + B_3 \left(\int_M v^m dV_g \right)^{\frac{1}{m}} \left(\int_M (\phi v)^{\frac{2m}{m-1}} dV_g \right)^{\frac{m-1}{m}}. \end{aligned}$$

Consequently

$$\{\Lambda(M, g) - B_3 \left(\int_M v^m dV_g \right)^{\frac{1}{m}}\} \left(\int_M (\phi v)^{\frac{2m}{m-1}} dV_g \right)^{\frac{m-1}{m}} \leq B_1 \int_M |\nabla \phi|^2 v^2 dV_g.$$

Under the assumption (4.11), we may choose sufficiently small ε_1 and ε_2 such that

$$\Lambda(M, g) - B_3 \left(\int_M v^m dV_g \right)^{\frac{1}{m}} > 0.$$

The remaining discussion is similar to that for Theorem 3.2. ■

Now let us look at the case that $R > 0$. By the solution of Yamabe problem, we know that the Yamabe constant $\Lambda(M, g)$ is attained by a positive function $u \in C^\infty(M)$. The metric $\tilde{g} = u^{\frac{4}{n-2}} g$ ($n = 2m$), called the Yamabe metric, has constant scalar curvature given by (cf. [He1], [LP]):

$$S_{\tilde{g}} = \frac{2(2m-1)}{m-1} \Lambda(M, g) \text{Vol}(\tilde{g})^{-\frac{1}{m}}. \quad (4.13)$$

It is known that any Einstein metric on a compact Riemannian n -manifold must be the Yamabe metric, provided it is not conformal to the standard metric of n -sphere ([Ob]). Since (M, g, J) is Kähler-Einstein, g is the Yamabe metric in its conformal class $[g]$. Hence (4.13) implies

$$\Lambda(M, g) = \frac{(m-1)R}{2m-1} \text{Vol}(M)^{\frac{1}{m}}. \quad (4.14)$$

As in §3, we give two types of rigidity results for this case. The first one is the following L^m -pinching result:

Theorem 4.3. *Let (M, g, J) be a compact Kähler-Einstein m -manifold with $m \geq 2$ and $R > 0$. Set*

$$Q(m) = \begin{cases} \frac{m(m+3)}{m+1}, & m = 2, 3 \\ \frac{2(2m-1)}{m-1}, & m \geq 4. \end{cases}$$

If

$$\|B\|_{L^m(M)} < \Lambda(M, g) Q(m) \frac{\sqrt{m^2-1}}{3(m^2-2)}, \quad (4.15)$$

then (M, g, J) is biholomorphically homothetic to the complex projective space CP^m .

Proof. By integrating (4.8) and using the Hölder inequality, we have

$$\begin{aligned} \frac{m+3}{m+1} \int_M |\nabla|B||^2 dV_g &\leq \frac{3(m^2-2)}{\sqrt{m^2(m^2-1)}} \|B\|_{L^m(M)} \left(\int_M |B|^{\frac{2m}{m-1}} dV_g \right)^{\frac{m-1}{m}} \\ &\quad - \frac{2R}{m} \int_M |B|^2 dV_g. \end{aligned} \quad (4.16)$$

Applying (2.2) to $|B|$ leads to

$$\Lambda(M, g) \left(\int_M |B|^{\frac{2m}{m-1}} dV_g \right)^{\frac{m-1}{m}} \leq \int_M |\nabla|B||^2 dV_g + \frac{(m-1)R}{2m-1} \int_M |B|^2 dV_g. \quad (4.17)$$

It follows from (4.16) and (4.17) that

$$\begin{aligned} \int_M |\nabla|B||^2 dV_g &\leq \frac{3(m+1)(m^2-2)||B||_{L^m(M)}}{\Lambda(M,g)(m+3)\sqrt{m^2-1}} \int_M \{|\nabla|B||^2 + \frac{(m-1)R}{2m-1}|B|^2\} dV_g \\ &\quad - \frac{2(m+1)R}{m(m+3)} \int_M |B|^2 dV_g. \end{aligned}$$

Consequently

$$\begin{aligned} &\left\{1 - \frac{3(m+1)(m^2-2)}{\Lambda(M,g)m(m+3)\sqrt{m^2-1}} ||B||_{L^m(M)}\right\} \int_M |\nabla|B||^2 dV_g \\ &+ \left\{\frac{2(m+1)}{m(m+3)} - \frac{3(m^2-2)\sqrt{m^2-1}}{\Lambda(M,g)m(m+3)(2m-1)} ||B||_{L^m(M)}\right\} R \int_M |B|^2 dV_g \leq 0. \end{aligned} \quad (4.18)$$

It is easy to verify that (4.15) implies that the two terms on the left hand side of (4.18) are nonnegative. This leads to $B = 0$, that is, (M, g, J) has constant holomorphic sectional curvature $\frac{2R}{m(m+1)} > 0$. Then Synge's theorem ensures that M is simply connected. Hence (M, g, J) is biholomorphically homothetic to the complex projective space CP^m with the Fubini-Study metric (cf. Theorem 7.9 in [KN], Vol.II). ■

Remark 4.1. In [IK], Itho and Kobayashi gave a similar L^m -pinching result to characterize CP^m . However, their pinching constant is an abstract number depending on n and R . Our pinching constant seems better and more explicit than theirs.

The next one is the following pointwise pinching result.

Theorem 4.4. Let (M, g, J) be a compact Kähler-Einstein m -manifold with $m \geq 2$ and $R > 0$. If

$$|B| < \frac{2\sqrt{m^2-1}R}{3(m^2-2)}, \quad (4.19)$$

then (M, g, J) is biholomorphically homothetic to the complex projective space CP^m .

Proof. From (4.6), we have

$$\begin{aligned} \frac{1}{2} \Delta |B|^2 &\geq |\nabla B|^2 + \left(\frac{2R}{m} - \frac{3(m^2-2)}{m\sqrt{m^2-1}} |B| \right) |B|^2 \\ &\geq \left(\frac{2R}{m} - \frac{3(m^2-2)}{m\sqrt{m^2-1}} |B| \right) |B|^2. \end{aligned} \quad (4.20)$$

Under the assumption (4.19), the integration of (4.20) implies immediately that $B = 0$. Hence (M, g, J) is biholomorphically homothetic to CP^m . ■

Remark 4.2. It is obvious that if the condition (4.19) is replaced by

$$|B| \leq \frac{2\sqrt{m^2-1}R}{3(m^2-2)}, \quad (4.21)$$

then either $B = 0$ or $|B| = \frac{2\sqrt{m^2-1}R}{3(m^2-2)}$ and $\nabla B = 0$. It would be interesting to investigate the case when the equality of (4.21) holds.

By combining Theorem 3.1 and Theorem 4.1, we get

Theorem 4.5. *Let (M, g, J) be a complete noncompact Kähler m -manifold ($m \geq 2$) with zero scalar curvature and positive Yamabe constant. If*

$$\sqrt{2}\|E\|_{L^m(M)} + \|B\|_{L^m(M)} < \frac{4\Lambda(M, g)(m^2 + 1)\sqrt{m^2 - 1}}{3m(m + 1)(m^2 - 2)},$$

then (M, g, J) has constant holomorphic sectional curvature 0. Furthermore, if M is simply connected, then (M, g, J) is biholomorphically isometric to C^m .

Proof. It is easy to verify that

$$\frac{4\Lambda(M, g)(m^2 + 1)\sqrt{m^2 - 1}}{3m(m + 1)(m^2 - 2)} < \frac{4\Lambda(M, g)(m^2 - m + 1)}{m^3} \sqrt{\frac{2(m + 1)(m + 2)}{2m^2 + 4m + 3}}$$

for $m \geq 2$. Then, by using Theorems 3.1, 4.1 successively, we may prove the assertions. \blacksquare

Since

$$\frac{(m + 3)\sqrt{m^2(m^2 - 1)}}{3(m + 1)(m^2 - 2)} < \frac{(m + 1)}{m} \sqrt{\frac{2(m + 1)(m + 2)}{2m^2 + 4m + 3}},$$

Theorems 3.2, 4.2 imply that

Theorem 4.6. *Let (M, g, J) be a complete noncompact Kähler m -manifold ($m \geq 4$) with constant negative scalar curvature and positive Yamabe constant. Suppose*

$$\int_{B_r} (|E|^2 + |B|^2) dV_g = o(r^2) \quad \text{as } r \rightarrow \infty.$$

If

$$\sqrt{2}\|E\|_{L^m(M)} + \|B\|_{L^m(M)} < \frac{\Lambda(M, g)(m + 3)\sqrt{m^2(m^2 - 1)}}{3(m + 1)(m^2 - 2)},$$

then (M, g, J) has constant holomorphic sectional curvature $\frac{2R}{m(m+1)}$.

One may verify that the pinching constant in Theorem 4.3 is smaller than that in Theorem 3.3. Likewise, we have

Theorem 4.7. *Let (M, g, J) be a compact Kähler m -manifold ($m \geq 2$) with constant positive scalar curvature. Let $Q(m)$ be as in Theorem 4.3. If*

$$\sqrt{2}\|E\|_{L^m(M)} + \|B\|_{L^m(M)} < \frac{\Lambda(M, g)Q(m)\sqrt{m^2 - 1}}{3(m^2 - 2)},$$

then (M, g, J) is biholomorphically homothetic to CP^m .

Finally, Theorem 3.4 and Theorem 4.4 lead to

Theorem 4.8. *Let (M, g, J) be a compact Kähler m -manifold ($m \geq 2$) with constant positive scalar curvature R . If*

$$\sqrt{2}|E| + |B| < \frac{2\sqrt{m^2 - 1}R}{3(m^2 - 2)},$$

then (M, g, J) is biholomorphically homothetic to CP^m .

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